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Analytic solutions for three- and four-wave mixing via generalised Bose operators

Jacob Katriel† and David G Hummer‡

Joint Institute for Laboratory Astrophysics, National Bureau of Standards and University of Colorado, Boulder, Colorado 80309 USA

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Abstract. New types of generalised Bose operators are constructed. They are applied to the linearisation of the equations of motion describing three- and four-wave mixing, resulting in integral equations for the temporal behaviour of the various fields. Some mathematical properties of these integral equations are studied, thus establishing the equivalence between approaches differing in the manner in which the linearisation is carried out. The integral equations are solved analytically in terms of Jacobian elliptic functions.

1. Introduction

The construction of generalised Bose operators which satisfy boson commutation relations but which, when acting on the states of some more fundamental bosons, create or destroy several of them simultaneously, has been undertaken by Brandt and Greenberg (1969). The possibility of simplifying the treatment of second harmonic generation (Katriel 1971) and of the anharmonicity in one-dimensional oscillators (Rasetti 1972) using these operators was considered some time ago. Rasetti (1972) also derived a closed form expression for the generalised Bose operators, which, apart from certain phase factors which are discussed below, is equivalent to the normal ordered infinite series of Brandt and Greenberg (1969). The relevance of these operators to the study of the dynamical symmetry groups of anisotropic multi-dimensional harmonic oscillators was clarified by the demonstration (Katriel and Adam 1971) that they are actually equivalent to the operators introduced by Demkov (1963) in that context. They were independently introduced in a study of the anisotropic oscillator by Louck *et al* (1973), in a form identical to that of Rasetti (1972).

We shall now consider the construction of two types of even more general Bose operators, simultaneously affecting the occupancy of different boson modes. Such operators will be presented and their applicability demonstrated in the context of nonlinear optics. They can also be useful in other contexts, such as the treatment of coupled anharmonic oscillators, which we shall not consider in detail.

To motivate some aspects of the introduction of the generalised Bose operators for different modes, we first reconsider the single-mode generalised Bose operators. This will give us an opportunity to improve on and define the limits of validity of some

† JILA Visiting Fellow, 1978–79, on leave of absence from the Department of Chemistry, Technion-Israel Institute of Technology, Haifa 32000, Israel.
‡ Staff Member, Quantum Physics Division, National Bureau of Standards.

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previous results. The most interesting new result with respect to the treatment of second harmonic generation using the single-mode generalised Bose operators is the derivation and solution of an integral equation for the temporal behaviour of the first and second harmonic fields.

We then construct generalised Bose operators creating an arbitrary number of different bosons. This is followed by the application of a combination of the Holstein–Primakoff and Schwinger relations to construct a class of generalised Bose operators, which simultaneously create a boson of one type and destroy a boson of a different type. The application of these operators to the linearisation of the sum-frequency generation Hamiltonian results in integral equations for the number of sum-frequency photons, whose forms depend on the manner in which the linearisation is carried out, but which are shown to have unique solutions, expressible in terms of Jacobian elliptic functions.

The linearisation of Hamiltonians describing four-wave mixing is considered in the last section. The various different approaches, consisting of using different types of generalised Bose operators, are shown to result in integral equations which form natural generalisations of those obtained for three-wave mixing. The equivalence of the different linearisation procedures is established and analytic solutions, in terms of elliptic functions, are presented.

2. Single-mode generalised Bose operators; second harmonic generation

Let

$$A = F(\hat{n})a^2, \qquad A^+ = (a^+)^2F^*(\hat{n}), \tag{1}$$

where $\hat{n} = a^+a$, and require that $[A, A^+] = 1$. One obtains, using the fact that $aF(\hat{n}) = F(\hat{n}+1)a$, the following identity:

$$|F(n)|^2(n+1)(n+2) - |F(n-2)|^2 n(n-1) = 1. \tag{2}$$

It is easy to see that $|F(0)|^2 = \frac{1}{2}$ and $|F(1)|^2 = \frac{1}{6}$. One can then prove by induction that

$$|F(n)|^2 = 1/[2n+3-(-1)^n]. \tag{3}$$

Using Wilcox's (1967) normal ordering expansion formula, one immediately obtains A in the infinite series form derived by Brandt and Greenberg (1969). In the applications we shall assume that $F(n)$ is real, which is equivalent to assuming that all the phases in the form of the generalised Bose operators given by Brandt and Greenberg vanish.

Using the generalised Bose operators in the form of equation (1), we can improve on a result given some time ago by one of the present authors (Katriel 1971) for the Hamiltonian

$$\mathscr{H} = \hbar\omega a^+a + \hbar 2\omega b^+b + \hbar\varepsilon[b^+a^2 + (a^+)^2b], \tag{4}$$

describing second harmonic generation. Writing $a^2 = [1/F(\hat{n})]A$, $(a^+)^2 = A^+[1/F(\hat{n})]$, we obtain the Hamiltonian

$$\mathscr{H} = 2\hbar\omega(A^+A + b^+b) + \hbar\varepsilon\left(b^+\frac{1}{F(\hat{n})}A + A^+\frac{1}{F(\hat{n})}b\right). \tag{5}$$

Assuming that $F(\hat{n})$ can be treated as a constant, and following the steps of Katriel (1971), we obtain

$$n_2(t) = (n/2)\sin^2(\tilde{\varepsilon}t) \tag{6}$$

where $\tilde{\varepsilon} = \varepsilon/F(n) \simeq \varepsilon(2n)^{1/2}$, n is the number of ω-frequency photons initially and $n_2(t)$ is the number of second harmonic photons.

This result suggests an oscillatory behaviour of the population in the second harmonic mode, with a time period of $T \simeq \pi/[\varepsilon(2n)^{1/2}]$ compared with a period proportional to $\ln(n)/\sqrt{n}$, obtained in an approximate quantum mechanical computation by Walls (1970). However, the present treatment can only be valid quantitatively as long as $n_2 \ll n/2$, i.e. for $t \ll T$. For such short times it gives $n_2 \simeq n^2\varepsilon^2 t^2$, in agreement with Walls and Tindle (1971, 1972). Moreover, the short-time form of

$$b^+(t) = e^{it2\omega}[b^+ \cos(\tilde{\varepsilon}t) + iA^+ \sin(\tilde{\varepsilon}t)], \tag{7}$$

i.e.

$$b^+(t) \simeq e^{it2\omega}[b^+ + i(a^+)^2\varepsilon t - \varepsilon^2 t^2 b^+(\hat{n}_a + 1)], \tag{8}$$

differs from the expression given by Mista and Perina (1977), who have treated the exact equations of motion in a short-time approximation, only in the fact that $\hat{n}_a + 1$ in the last term should be replaced by $2\hat{n}_a + 1$.

To improve the model further we have to account for the fact that $\hat{n}$ in $F(\hat{n})$ is not a constant. Substituting $n(t)$ for $\hat{n}$, we obtain the following explicitly time-dependent Hamiltonian,

$$\mathcal{H} = 2\hbar\omega(A^+A + b^+b) + \hbar\varepsilon\{2[n(t)+1]\}^{1/2}(b^+A + A^+b). \tag{9}$$

Introducing $X^+ \equiv (b^+ + A^+)/\sqrt{2}$, $Y^+ = (b^+ - A^+)/\sqrt{2}$, the Hamiltonian becomes

$$\mathcal{H} = \hbar[\![2\omega + \varepsilon\{2[n(t)+1]\}^{1/2}]\!]X^+X + \hbar[\![2\omega - \varepsilon\{2[n(t)+1]\}^{1/2}]\!]Y^+Y. \tag{10}$$

From the equation of motion

$$i\hbar\dot{X} = [X, \mathcal{H}] = \hbar[\![2\omega + \varepsilon\{2[n(t)+1]\}^{1/2}]\!]X \tag{11}$$

we obtain

$$X(t) = X \exp\left[-i\left(2\omega t + \varepsilon\sqrt{2}\int_0^t (n(t)+1)^{1/2}\,dt\right)\right] \tag{12}$$

and a similar expression for $Y(t)$. Finally,

$$b^+(t) = e^{i2\omega t}\left[b^+ \cos\left(\varepsilon\sqrt{2}\int_0^t (n(t)+1)^{1/2}\,dt\right) + iA^+ \sin\left(\varepsilon\sqrt{2}\int_0^t (n(t)+1)^{1/2}\,dt\right)\right] \tag{13}$$

and

$$n_2(t) = \langle n|b^+(t)b(t)|n\rangle = (n/2)\sin^2\left(\varepsilon\sqrt{2}\int_0^t (n(t)+1)^{1/2}\,dt\right). \tag{14}$$

Hence the following integral equation is obtained for $n(t)$:

$$n(t) = n\cos^2\left(\varepsilon\sqrt{2}\int_0^t (n(t)+1)^{1/2}\,dt\right). \tag{15}$$

The solution of this equation is shown in appendix 1 to be

$$n(t) = n\,\mathrm{cn}^2\{\varepsilon[2(n+1)]^{1/2}t|n/(1+n)\}. \tag{16}$$

This result is almost identical to that given by Walls (1970), who used an approximation due to Bonifacio and Preparata (1970). It is, however, worthwhile noticing that equation (15) is very convenient for a numerical evaluation of $n(t)$. Moreover, the integral equation very transparently exhibits some of the qualitative features of the solution, which were derived by Walls (1970) from his analogue of equation (16). These include the fact that $n(t)$ varies periodically between n and zero, and that the system spends most of its time with $n(t)$ close to zero, which means that the second harmonic field, once generated, transforms to the original field at a considerably slower rate than its rate of formation. The time (τ) that it takes for half of the second harmonic photons to be generated is in good agreement with the expression obtained by assuming that $F(\hat{n})$ is a constant, i.e. $4\varepsilon\tau \simeq \pi/(2n)^{1/2}$.

Second harmonic generation was treated classically, via Maxwell's equations, by Armstrong *et al* (1962). Their result can be obtained by making the approximation $n+1 \simeq n$, as a consequence of which equation (16) reduces to

$$n(t) = n\ \mathrm{cn}^2[\varepsilon(2n)^{1/2}t|1] = n\ \mathrm{sech}^2(\varepsilon(2n)^{1/2}t),$$

which is a monotonically decreasing function of t. The difference between the classical and quantum mechanical results can be interpreted by noting that quantum mechanically at least zero point vibrations of both the fundamental and second harmonic waves always exist, which implies that the classical initial conditions are not quite realised. This difference demonstrates that the present treatment retains the essential quantum mechanical features of the problem, while at the same time it achieves an almost classical mathematical simplicity. The same remarks apply to the treatment of further nonlinear phenomena, and will not be repeated.

The generalisation to boson operators involving k bosons

$$A^{(k)} = F_k(\hat{n})a^k, \qquad A^{(k)+} = (a^+)^k F_k^*(\hat{n}), \tag{17}$$

is straightforward. From the commutation relation $[A^{(k)}, A^{(k)+}] = 1$ we obtain the identity

$$|F_k(n)|^2(n+k)!/n! - |F_k(n-k)|^2 n!/(n-k)! = 1,$$

which can be used to prove by induction that

$$|F_k(n)|^2 = ([n/k]+1)n!/(n+k)! \tag{18}$$

where $[x]$ is the largest integer smaller than x. Again, equivalence with the form derived by Brandt and Greenberg (1969) is easily established.

Equation (18) was derived by Rasetti (1972), who obtained it without the absolute value sign. This means that his result corresponds to the assumption that all the phases present in the Brandt and Greenberg form of the generalised Bose operator vanish.

3. Generalised Bose operators for different modes: sum-frequency generation

An even more interesting result will now be derived for the pair of operators

$$C = F(\hat{n}_a, \hat{n}_b)ab, \qquad C^+ = b^+a^+F^*(\hat{n}_a, \hat{n}_b), \tag{19}$$

where a and b are boson operators for two different types of bosons and $\hat{n}_a$, $\hat{n}_b$ are the corresponding number operators. For a proper choice of $F(\hat{n}_a, \hat{n}_b)$ the operators C and C^+ satisfy a boson commutation relation, but can be considered as destruction and creation operators for a pair of different bosons. From the requirement $[C, C^+]=1$ we obtain

$$|F(n_a, n_b)|^2(n_a+1)(n_b+1)-|F(n_a-1, n_b-1)|^2 n_a n_b=1. \tag{20}$$

One can show by induction that

$$|F(n_a, n_b)|^2=1/(n_>+1) \tag{21}$$

where $n_> = \max(n_a, n_b)$.

The expressions for C and C^+ as normal ordered expansions in a^+, a, b^+ and b can be obtained by a straightforward double application of Wilcox's (1967) formula.

The operators C and C^+ seem to be particularly useful and appealing when a and b are two modes which interact nonlinearly, but the intensity of one of them is much higher than that of the other. In this case the higher-intensity mode can be assumed to be almost unaffected by the interaction. Consequently, $F(\hat{n}_a, \hat{n}_b)$, which depends only on this higher-intensity mode, can be treated as a constant. This results in a considerable simplification of the problem, as will be demonstrated in detail.

Consider the Hamiltonian (Gambini 1977, and references therein)

$$\mathcal{H}=\hbar\omega_a a^+a+\hbar\omega_b b^+b+\hbar\omega_c c^+c+\hbar\varepsilon(c^+ab+a^+b^+c) \tag{22}$$

where $\omega_c=\omega_a+\omega_b$, describing sum-frequency generation. Notice that

$$a^+a|n_a, n_b\rangle=n_a|n_a, n_b\rangle, \qquad b^+b|n_a, n_b\rangle=n_b|n_a, n_b\rangle$$

and

$$\begin{aligned} C^+C|n_a, n_b\rangle &= b^+a^+|F(\hat{n}_a, \hat{n}_b)|^2 ab|n_a, n_b\rangle \\ &= (n_a n_b/n_>)|n_a, n_b\rangle = n_<|n_a, n_b\rangle. \end{aligned}$$

As $\hat{n}_a-\hat{n}_b$ is a constant of the motion we can write, assuming that a is the higher-intensity mode,

$$b^+b=C^+C, \qquad a^+a=n_0+C^+C,$$

where n_0 is the difference in the number of photons in the two modes. Hence

$$\mathcal{H}=\hbar\omega_a n_0+\hbar\omega_c(C^+C+c^+c)+\hbar\varepsilon\left(c^+\frac{1}{F(\hat{n}_a, \hat{n}_b)}C+C^+\frac{1}{F^*(\hat{n}_a, \hat{n}_b)}c\right). \tag{23}$$

Subject to the assumption that $n_a \gg n_b$, which is the situation usually encountered in practice (Mista and Perina 1977), we can write

$$\mathcal{H}'=\hbar\omega_c(C^+C+c^+c)+\hbar\tilde{\varepsilon}(c^+C+C^+c) \tag{24}$$

where $\tilde{\varepsilon}=\varepsilon/F \simeq \varepsilon\sqrt{n_a}$, and where the constant term $\hbar\omega_a n_0$ has been dropped. From the Heisenberg equation of motion we obtain

$$c^+(t)=e^{i\omega_c t}[C^+ i\sin(\tilde{\varepsilon}t)+c^+\cos(\tilde{\varepsilon}t)]. \tag{25}$$

The number of photons with the sum frequency is given by

$$N_c(t)=\langle n_a, n_b|c^+(t)c(t)|n_a, n_b\rangle=\sin^2(\tilde{\varepsilon}t)n_<, \tag{26}$$

exhibiting oscillations with a periodicity $T=\pi/\tilde{\varepsilon}=\pi/(\varepsilon\sqrt{n_a})$.

Solving the exact equations of motion numerically, Walls and Barakat (1970) obtained an oscillatory behaviour of the number of sum-frequency photons for $n_a \gg n_b$.

A period of $\sim 0.51/\varepsilon$ can be estimated from their plot for $n_a = 50$, $n_b = 24$, compared with $\sim 0.44/\varepsilon$ obtained in our treatment. Actually, by taking for n_a the average between its maximum value of 50 and its minimum value of 26, we obtain the recurrence time $\sim 0.507/\varepsilon$, in very close agreement with Walls and Barakat (1970).

For times which are short relative to T, the above result should hold even if the intensities in the two modes do not differ considerably, simply because only a small fraction of the photons has been depleted by the interaction. Under such conditions

$$c^+(t) \simeq e^{i\omega_c t}[c^+ + i\varepsilon t b^+ a^+ - (\varepsilon^2 t^2/2)c^+(n_> + 1)]. \tag{27}$$

This result differs from that given by Agrawal and Mehta (1974) on the basis of a short-time treatment of the exact equations of motion only by the fact that the factor $(n_> + 1)$ in the third term has to be substituted by $(n_> + n_< + 1)$. If $n_< \ll n_>$ the two treatments become identical at short times but the present one, in its unexpanded form, is valid for longer times as well.

If we replace $\hat{n}_a$ in equation (23) by $n_a(t)$, the following integral equation can be derived for $n_a(t)$,

$$n_a(t) = n_a - n_b \sin^2\left(\varepsilon \int_0^t (n_a(t))^{1/2}\, dt\right). \tag{28}$$

This equation can be solved in a manner similar to equation (15), obtaining

$$n_a(t) = n_a - n_b\, \mathrm{sn}^2[\varepsilon \sqrt{n_a} t | (n_b/n_a)]. \tag{29}$$

This result is very similar to that obtained for an equivalent problem by Bonifacio and Preparata (1970). Again the integral equation, equation (28), is very convenient for numerical evaluation of $n(t)$ as well as for analysing its qualitative properties. It follows very transparently from equation (28) that $n_a(t)$ is periodic, spending more time in states with close to minimal numbers of a (and b) type photons, than in states in which the c field intensity is close to minimal.

Using the integral equation, one can generate the expansion of $n_a(t)$ in powers of t^2, obtaining

$$\begin{aligned} n_a(t) = n_a - \varepsilon^2 n_a n_b t^2 + \varepsilon^4 n_a n_b (n_a + n_b) t^4/3 - \varepsilon^6 n_a n_b (13 n_a n_b + 2n_a^2 + 2n_b^2) t^6/45 \\ + \varepsilon^8 n_a n_b [n_a^3 + n_b^3 + 30 n_a n_b (n_a + n_b)] t^8/315 - \ldots \end{aligned} \tag{30}$$

The first two terms in this expansion are identical to the exact results of Scharf (1974), and the other three differ from his by quantities which are of the order of $1/n_a$ (or $1/n_b$).

4. Generalised Bose operators for several different modes

Whereas the single-mode generalised Bose operators were constructed both by Brandt and Greenberg (1969) and by Rasetti (1972) so as to create an arbitrary number of bosons, the different modes operators introduced in § 3 referred to two bosons only. However, the generalisation to an arbitrary number of different bosons is straightforward.

Consider the product $a^\dagger b^\dagger c^\dagger$. Introducing the two-mode generalised Bose operator

$$C^\dagger = a^\dagger b^\dagger [\max(n_a, n_b) + 1]^{-1/2} \tag{31}$$

we obtain

$$a^{\dagger}b^{\dagger}c^{\dagger} = C^{\dagger}c^{\dagger}[\max(n_a, n_b)+1]^{1/2}.$$

Using the two-mode reduction with respect to $C^{\dagger}c^{\dagger}$, we obtain

$$X^{\dagger} = C^{\dagger}c^{\dagger}[\max(n_c, n_C)+1]^{-1/2}.$$

However, from equation (31) it follows that

$$n_C = \min(n_a, n_b).$$

Hence

$$X^{\dagger} = a^{\dagger}b^{\dagger}c^{\dagger}\{[\max(\hat{n}_a, \hat{n}_b)+1][\max(\hat{n}_c, \min(\hat{n}_a, \hat{n}_b))+1]\}^{-1/2}.$$

It is easy to show that the coefficient can easily be written in a more symmetrical form, so that finally

$$X^{\dagger} = a^{\dagger}b^{\dagger}c^{\dagger}\left(\frac{\min(\hat{n}_a, \hat{n}_b, \hat{n}_c)+1}{(\hat{n}_a+1)(\hat{n}_b+1)(\hat{n}_c+1)}\right)^{1/2}. \qquad (32)$$

One can similarly derive the generalised Bose operator creating four different modes,

$$Y^{\dagger} = a^{\dagger}b^{\dagger}c^{\dagger}d^{\dagger}\left(\frac{\min(\hat{n}_a, \hat{n}_b, \hat{n}_c, \hat{n}_d)+1}{(\hat{n}_a+1)(\hat{n}_b+1)(\hat{n}_c+1)(\hat{n}_d+1)}\right)^{1/2}. \qquad (33)$$

The generalisation to an arbitrary number of different bosons is now obvious. The commutation relation $[Y, Y^{\dagger}]=1$ is easily verified.

5. An additional type of generalised Bose operator

A representation of angular momentum operators by means of products of a boson creation operator and a boson destruction operator was constructed by Schwinger (1965). His relation is

$$a^{\dagger}b = J_+, \qquad b^{\dagger}a = J_-, \qquad (\hat{n}_a - \hat{n}_b)/2 = J_z.$$

A relation between angular momentum operators and a single boson operator was constructed by Holstein and Primakoff (1940), who showed that

$$J_+ = (2J - \hat{n}_c)^{1/2}c. \qquad (34)$$

Combining these relations, we obtain

$$a^{\dagger}b = (2J - \hat{n}_c)^{1/2}c \qquad (35)$$

and

$$b^{\dagger}a = c^{\dagger}(2J - \hat{n}_c)^{1/2}.$$

Using Schwinger's relation, it is easy to see that

$$\hat{J}^2 = (\hat{n}/2)(\hat{n}/2+1)$$

where

$$n = \hat{n}_a + \hat{n}_b.$$

This result indicates that the angular momentum quantum number J, in terms of which the eigenvalue of $\hat{J}^2$ is $J(J+1)$, is given by $J=(n_a+n_b)/2$. Note that $\hat{n}$ commutes with $a^\dagger b$ and $b^\dagger a$, and therefore is a constant of the motion.

From equation (35) it follows that the state with the maximum value of n_c, which is according to equation (34) equal to $2J=n_a+n_b$, corresponds to $n_a=0$ and a maximum value of n_b, i.e. $n_b=2J$. The possible values of n_c are

$$n_c=0, 1, \ldots, (n_a+n_b)$$

and from the correspondence between $n_c=2J$ and $n_b=2J$ and equation (35) it follows that for all states $n_c=n_b$.

6. Equivalence between different treatments of three-wave mixing

The Hamiltonian describing sum-frequency generation, equation (22), was studied in § 3 using the two-mode generalised Bose operator $C^\dagger=a^\dagger b^\dagger/(n_>+1)^{1/2}$.

We can now consider linearising this Hamiltonian by means of the generalised Bose operators related to either $c^\dagger a$ or $c^\dagger b$. Using the former, we obtain a result which is identical to equation (28). However, using the latter we obtain

$$n_c(t)=n_a \sin^2\left(\varepsilon \int_0^t [n_b-n_c(t)]^{1/2}\, \mathrm{d}t\right). \tag{36}$$

From the result in § 3, it follows that the solution of (28) can be written as

$$n_c(t)=n_b \operatorname{sn}^2(\varepsilon\sqrt{n_a}t \,|\, n_b/n_a). \tag{37}$$

Similarly, the solution of (36) is

$$n_c(t)=n_a \operatorname{sn}^2(\varepsilon\sqrt{n_b}t \,|\, n_a/n_b). \tag{38}$$

The equivalence between (37) and (38) follows from Jacobi's real transformation (Abramowitz and Stegun 1967). A somewhat subtle aspect of the equivalence between equations (28) and (36) is discussed in appendix 2.

7. Four-wave mixing

In view of the large variety of different four-wave phenomena we shall treat only two representative cases in detail, so as to exhibit the main features of the linearisation of the corresponding Hamiltonians in terms of the generalised Bose operators, and of the analytic solution of the resulting integral equations.

The Hamiltonian

$$\mathcal{H}=\sum_{i=1}^{4} \hbar\omega_i+\hbar\varepsilon(a_1^\dagger a_2^\dagger a_3 a_4+a_3^\dagger a_4^\dagger a_1 a_2) \tag{39}$$

can be reduced in several different ways, using different forms of generalised Bose operators. Let us consider the initial conditions $n_1>n_2>0$, $n_3=n_4=0$, and let

$$a_1^\dagger a_2^\dagger=C^\dagger(\hat{n}_1+1)^{1/2}, \qquad a_3^\dagger a_4^\dagger=D^\dagger(\hat{n}_3+1)^{1/2}.$$

Following the steps of the derivation in § 3, we obtain

$$n_2(t)=n_2\cos^2\left(\varepsilon\int_0^t[(n_1-n_2+n_2(t)+1)(n_2-n_2(t)+1)]^{1/2}\,\mathrm{d}t\right). \tag{40}$$

The solution of this equation is shown in appendix 1 to be

$$n_2(t)=n_2(n_2+1)\,\mathrm{cn}^2(t/\tau\,|\,k^2)/[1+n_2\,\mathrm{cn}^2(t/\tau\,|\,k^2)], \tag{41}$$

where

$$\tau=\varepsilon^{-1}[(n_1+1)(n_2+1)]^{-1/2} \tag{42}$$

and

$$k^2=\frac{n_2}{n_2+1}\,\frac{n_1+2}{n_1+1}. \tag{43}$$

A Hamiltonian which is closely related to that given by equation (39) describes the recently extensively studied process of CARS (for a recent review see Swofford and Albrecht 1978).

Let the interaction be

$$\mathscr{H}=\hbar\varepsilon[(a_1^\dagger)^2a_2b+a_2^\dagger b^\dagger a_1^2] \tag{44}$$

and the initial conditions

$$n_1>0, \qquad n_2>0, \qquad n_b=0.$$

Linearisation of this Hamiltonian is achieved with

$$A^\dagger=(a_1^\dagger)^2[2(n_1+1)]^{-1/2}, \qquad B^\dagger=a_2^\dagger b^\dagger(n_2+1)^{-1/2},$$

and the integral equation finally obtained is

$$n_1(t)=n_1\cos^2\left(\varepsilon\int_0^t[(n_1(t)+1)(2n_2+n_1+2-n_1(t))]^{1/2}\,\mathrm{d}t\right). \tag{45}$$

The equation is solved in the same way as was equation (41) to obtain

$$n_1(t)=n_1(2n_2-2+n_1)\,\mathrm{cn}^2(t/\tau\,|\,k^2)/[n_1\,\mathrm{cn}^2(t/\tau\,|\,k^2)+2n_2-2], \tag{46}$$

where

$$\tau=\varepsilon^{-1}[(n_1+1)(2n_2+n_1-2)]^{-1/2} \tag{47}$$

and

$$k^2=n_1(n_1+2n_2-1)/(1+n_1)(n_1+2n_2-2). \tag{48}$$

A second major group of four-wave phenomena is described by the Hamiltonian

$$\mathscr{H}=\sum_{i=1}^4\hbar\omega_i a_i^\dagger a_i+\hbar\varepsilon(a_1^\dagger a_2a_3a_4+a_2^\dagger a_3^\dagger a_4^\dagger a_1). \tag{49}$$

For the initial conditions

$$n_1=0, \qquad 0<n_2\leqslant n_3\leqslant n_4,$$

we linearise by means of

$$A^\dagger=[n_3(t)\cdot n_4(t)]^{-1/2}a_2^\dagger a_3^\dagger a_4^\dagger$$

and obtain

$$n_1(t) = n_2 \sin^2\left(\varepsilon \int_0^t [(n_3 - n_1(t))(n_4 - n_1(t))]^{1/2} \,dt\right). \tag{50}$$

Alternatively, we linearise by means of

$$a_2^\dagger a_1 = [n_2 - n_1(t)]^{1/2} A, \quad \cdot \quad a_3 a_4 = B[n_4(t)]^{1/2},$$

obtaining

$$n_1(t) = n_3 \sin^2\left(\varepsilon \int_0^t [(n_2 - n_1(t))(n_4 - n_1(t))]^{1/2} \,dt\right). \tag{51}$$

The discussion in appendix 2 shows that (50) and (51) are equivalent, provided that the sign of the square root in the integrand is properly chosen. Equation (51) is solved by the method outlined in appendix 1 to obtain

$$n_1(t) = n_2 \,\mathrm{sn}^2(t/\tau \,|\, k^2)/[1 - (n_2/n_4)\,\mathrm{cn}^2(t/\tau \,|\, k^2)], \tag{52}$$

where

$$\tau = \varepsilon^{-1}[(n_4 - n_2)n_3]^{-1/2} \tag{53}$$

and

$$k^2 = (n_4 - n_3)n_2/(n_4 - n_2)n_3. \tag{54}$$

8. Discussion

The application of the single-mode and different-modes generalised Bose operators to nonlinear optics has been demonstrated to result in a very significant mathematical simplification, as well as a very transparent physical interpretation of each stage of the treatment. It also yields useful and convenient forms of the results, such as the integral equations for the intensities of the different frequency fields. It is particularly noteworthy that the application of generalised Bose operators results in a treatment of four- (and, in principle, higher-) wave mixing in a manner which is an obvious generalisation of that for three-wave mixing, without major new difficulties. We note that in all cases some arbitrariness exists because of relations such as

$$a^\dagger b^\dagger (\hat{n}_> + 1)^{-1/2} = (\hat{n}_>)^{-1/2} a^\dagger b^\dagger. \tag{55}$$

Clearly, in most cases of interest $n_> \gg 1$ and no real problem arises. Otherwise, it may be reasonable, when introducing the approximation $\hat{n}_> = n_>(t)$, to write

$$a^\dagger b^\dagger = C^\dagger [n_>(t) + \tfrac{1}{2}]^{1/2},$$

which corresponds to taking an intermediate value between the two forms resulting from equation (55).

It may be of interest to consider the possibility of improving on the present results by treating quantities such as $\hat{n} - n(t)$ as perturbations. From a physical point of view the most important next step would be the introduction of loss mechanisms within the framework of the generalised Bose operators, so as to allow damping of the oscillations we have so far been considering. The temporal (or spatial, along the optical path) intensity modulations predicted for four-wave mixing will, we hope, be confirmed by the rapidly developing four-wave spectroscopy.

Appendix 1.

Equation (15) can be solved by rewriting it in the form

$$n(t) = n_0 \cos^2 \theta(t) \tag{A1.1}$$

where

$$\theta(t) = \mu \int_0^t (n(t') + 1)^{1/2} \, \mathrm{d}t'. \tag{A1.2}$$

Differentiating (A1.2) and substituting (A1.1), we obtain

$$\mathrm{d}t = \mathrm{d}\theta / [\![\mu(1 + n_0)^{1/2} \{1 - [(n_0/(1 + n_0)] \sin^2 \theta\}^{1/2}]\!]. \tag{A1.3}$$

Integrating (A1.3), we obtain

$$\mu(1 + n_0)^{1/2} t = \int_0^\theta \mathrm{d}\theta' / \{1 - [n_0/(1 + n_0)] \sin^2 \theta'\}^{1/2}$$

or (Abramowitz and Stegun 1967)

$$\mathrm{cn}[\mu(1 + n_0)^{1/2} t \,|\, n_0/(1 + n_0)] = \cos \theta, \tag{A1.4}$$

i.e.

$$n(t) = n_0 \, \mathrm{cn}^2[\mu(1 + n_0)^{1/2} t \,|\, n_0/(1 + n_0)], \tag{A1.5}$$

where $\mathrm{cn}(x \,|\, m)$ is the Jacobian elliptic function. From the known periodicity of elliptic functions, we see that $n(t)$ has the period

$$p = 2K[n_0/(1 + n_0)] / \varepsilon(1 + n_0)^{1/2}, \tag{A1.6}$$

where

$$K(m) = \int_0^{\pi/2} \mathrm{d}\phi / (1 - m \sin^2 \phi)^{1/2}$$

is the complete elliptic integral of the first kind.

Equation (41) for four-wave mixing can be solved in much the same way, by writing it as

$$n_2(t) = n_2 \cos^2[\theta(t)] \tag{A1.7}$$

where

$$\theta(t) \equiv \varepsilon \int_0^t \mathrm{d}t' \, [(n_2(t') + \delta + 1)(1 + n_2 - n_2(t'))]^{1/2}. \tag{A1.8}$$

We have defined

$$\delta = n_1 - n_2 \, (>1). \tag{A1.9}$$

Differentiating (A1.8) with respect to t, substituting (A1.7) and then integrating, we obtain

$$2n_2 \varepsilon t = \int_0^{\sin^2 \theta} \mathrm{d}u \left[u(u-1)\left(u - \frac{1 + \delta + n_2}{n_2}\right)\left(u + \frac{1}{n_2}\right) \right]^{-1/2}. \tag{A1.10}$$

This integral can be reduced to an elliptic integral by the transformation

$$\sin\phi = [(n_2+1)u/(un_2+1)]^{1/2}; \tag{A1.11}$$

we obtain

$$\frac{t}{\tau} = \int_0^{\phi_1} d\phi/(1-k^2\sin^2\phi)^{1/2} = F(\phi_1|k^2) \tag{A1.12}$$

where

$$\tau = \varepsilon^{-1}[(1+n_2+\delta)(1+n_2)]^{-1/2}, \tag{A1.13}$$

$$k^2 = \frac{n_2(2+\delta+n_2)}{(1+\delta+n_2)(n_2+1)} = \frac{[n_2+(n_2/1+n_2)(1+\delta)]}{n_2+(1+\delta)}, \tag{A1.14}$$

and $\sin\phi_1$ is obtained by setting $u=\sin^2\theta$ in (A1.11) (cf Gröbner and Hofreiter (1965) for details). From the second expression in (A1.14) we see that $k^2 \leqslant 1$. From (A1.12) we have directly

$$\mathrm{sn}(t/\tau|k^2) = \sin\phi_1 \equiv [(n_2+1)\sin^2\theta/(n_2\sin^2\theta+1)]^{1/2}. \tag{A1.15}$$

Solving for $\sin^2\theta$ and substituting the resulting expression in (A1.7), we obtain the solution

$$n_2(t) = n_2(n_2+1)\,\mathrm{cn}^2(t/\tau|k^2)/[1+n_2\,\mathrm{cn}^2(t/\tau|k^2)]. \tag{A1.16}$$

The period of $n_2(t)$ is

$$p = 2\tau K(k^2). \tag{A1.17}$$

Appendix 2.

Consider the integral equation

$$f(t) = a\sin^2\left(\int_0^t [b-f(t)]^{1/2}F[t,f(t)]\,\mathrm{sgn}\left(\frac{df}{dt}\right)\cdot dt\right) \tag{A2.1}$$

where a and b are real and positive constants, $F[t, f(t)]$ is positive definite but otherwise arbitrary, and

$$\mathrm{sgn}(x) = \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0. \end{cases}$$

Differentiating with respect to t, we obtain

$$df/dt = 2a\sin\{\ldots\}\cos\{\ldots\}[b-f(t)]^{1/2}F[t,f(t)]\,\mathrm{sgn}(df/dt)$$

which, using (A2.1), becomes

$$df/dt = 2[f(t)]^{1/2}\{[a-f(t)][b-f(t)]\}^{1/2}F[t,f(t)]\,\mathrm{sgn}(df/dt). \tag{A2.2}$$

Equation (A2.2) is symmetrical with respect to interchange of a and b. To show that equation (A2.1) is also symmetrical with respect to interchange of a and b, all that has to be pointed out is that this interchange does not affect the initial condition $f(0)=0$.

If $a \leqslant b$, it is clear that $f(t)$ increases from 0 to a while t increases from 0 to T, where

$$\int_0^T [b-f(t)]^{1/2}F[t,f(t)]\,dt = \pi/2. \tag{A2.3}$$

The factor $\mathrm{sgn}(\mathrm{d}f/\mathrm{d}t)=1$ in this range and was therefore suppressed in equation (A2.3). For $T<t<2T$, $f(t)$ decreases and equation (A2.1) can be written in the form

$$f(t)=a\sin^2\left(\frac{\pi}{2}-\int_T^t\ldots\,\mathrm{d}t\right)=a\sin^2\left(\frac{\pi}{2}+\int_T^t\ldots\,\mathrm{d}t\right),$$

which shows that the factor $\mathrm{sgn}(\mathrm{d}f/\mathrm{d}t)$ is redundant in this range as well.

However, if we interchange a and b the value of the integral now appearing as the argument is, at $t=T$ (where $f(t)$ obtains its maximum value of a),

$$\int_0^T[a-f(t)]^{1/2}F[t,f(t)]\,\mathrm{d}t=\sin^{-1}(a/b)<\frac{\pi}{2}.$$

Clearly, the factor $\mathrm{sgn}(\mathrm{d}f/\mathrm{d}t)$ is not redundant, for $T<t<2T$, in this case. This factor guarantees that the square root in the integrand is correctly continued beyond the point $t=T$, at which it vanishes.

In the special case studied in § 6, i.e. $F[t,f(t)]=1$, the phase factor $\mathrm{sgn}(\mathrm{d}f/\mathrm{d}t)$ was not included, yet equivalence with respect to interchange of a and b was established. This is a consequence of the fact that the Jacobian elliptic functions, whose properties were used in that context, guarantee that the correct analytic continuation is made. Introducing that factor explicitly in equation (38) (with $n_a>n_b$), we obtain

$$n_c(t)=n_a\sin^2\left[\varepsilon\int_0^t[n_b-n_c(t)]^{1/2}\,\mathrm{sgn}\left(\frac{\mathrm{d}n_c}{\mathrm{d}t}\right)\mathrm{d}t\right].$$

When this equation is solved as in appendix 1 we get

$$n_c(t)=n_a\,\mathrm{sn}^2\left[\varepsilon(n_b)^{1/2}\int_0^t\mathrm{sgn}\left(\frac{\mathrm{d}n_c}{\mathrm{d}t}\right)\mathrm{d}t\,\middle|\,n_a/n_b\right].\qquad\text{(A2.4)}$$

Now, for $0<t<T$,

$$\int_0^t\mathrm{sgn}\left(\frac{\mathrm{d}n_c}{\mathrm{d}t}\right)\mathrm{d}t=t,$$

so that this result reduces, as it should, to equation (40). For $T<t<2T$

$$\int_0^t\mathrm{sgn}\left(\frac{\mathrm{d}n_c}{\mathrm{d}t}\right)\mathrm{d}t=T-(t-T)=2T-t$$

for which equation (A2.4) does not look identical to (40), but is in fact identical to it due to the periodicity properties of the Jacobian function.

As a practical recommendation it seems desirable to choose the smaller of a and b to be the coefficient in front of the $\sin^2\{\ldots\}$ in equation (A2.1). This allows us to drop the phase factor $\mathrm{sgn}(\mathrm{d}f/\mathrm{d}t)$ in the integrand and also results in an equation which can be more accurately numerically integrated because the integrand varies within narrower limits than with the other choice of parameter ordering.

It may further be noted that by integrating equation (A2.2) we obtain

$$f(t)=2\int_0^t\{f(t)[a-f(t)][b-f(t)]\}^{1/2}F[t,f(t)]\,\mathrm{sgn}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\mathrm{d}t.\qquad\text{(A2.5)}$$

This is also an integral equation for $f(t)$, which may have certain practical advantages over equation (A.2.1). For one, it enables a significant simplification in the generation of the power series expansion of $f(t)$. Equation (A2.5) can be transformed into a differential equation, which for $F[t, f(t)] = 1$ is

$$\mathrm{d}^2 f(t)/\mathrm{d}t^2 = 2ab - 4(a+b)f(t) + 6f(t)^2$$

with the initial conditions $f(t) = 0$, $\mathrm{d}f/\mathrm{d}t|_0 = 0$. Similarly, for $F[t, f(t)] = [c - f(t)]^{1/2}$, which is of interest in the context of four-wave mixing, we obtain

$$\mathrm{d}^2 f(t)/\mathrm{d}t^2 = 2abc - 4(ab + bc + ca)f + 6(a+b+c)f^2 - 8f^3,$$

with the same initial conditions.

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